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Exploiting Partial Symmetry in Vortex Lattice Models

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ARO, Inc.

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Symmetry is commonly used to cut computation time; however, it is usually an all-or-nothing proposition. This need not be the case for vortex lattice computations. As an example, consider an airplane, symmetric with respect to a vertical plane through its nose and tail. If the free stream is parallel to the plane of symmetry, then numerical computations can be applied to half the

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20. ABSTRACT (Continued)

airplane, and symmetry can be assumed for the other half. If, however, the free stream is not parallel to the plane of symmetry and/or if an instrument pod is added under one wing, then symmetry is spoiled. In vortex lattice computations, the symmetry of the airplane can still be used to advantage, not only in computation of the coefficients of the linear algebraic system, but more so in its solution. Computation time for a completely symmetric system is reduced, in comparison to that for an asymmetric system of the same size, by a factor of eight for one plane of symmetry. If there are two or three planes of symmetry, the factors are 64 and 512, respectively. If the system is only partially symmetric, the corresponding factors may still be as high as 4, 16, and 64.

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The results of the research presented were obtained by ARO, Inc., AEDC Group (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee, under ARO Project Number P43T-07. The manuscript was submitted for publication on November 7, 1980.

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APPENDIX

1.0 INTRODUCTION

The potential flow computer program (PFP) described in Ref. 1 has been used frequently for aerodynamic analyses in support of testing at the Arnold Engineering Development Center (AEDC). In one particular application, the investigation of the gun gas ingestion for the A-10, the airplane was symmetric but the gun was off center. The completely symmetric model required four hours of IBM 370/165 computer time to run on the PFP. To obtain a solution with the gun off center would have required about thirty hours per run. This rush project motivated a quick analysis of symmetry properties which resulted in a procedure whereby the off-center runs could be made in seven hours. About ten such runs were made. After the project was completed, a general method was developed as a spin-off under a computational fluid dynamics project.

Symmetry is commonly used to cut computation time; however, it is usually an all-or-nothing proposition. This need not be the case for vortex lattice computations. If the model has a nonsymmetric portion and/or if the boundary conditions are not symmetric, the symmetry of the system can still be used to advantage, not only in computation of the coefficients of the linear algebraic system, but also in its solution. Computation time for a completely symmetric system is reduced, compared to that for an asymmetric system of the same size, by a factor of eight for one plane of symmetry. If there are two or three planes of symmetry, the factors are 64 and 512, respectively. If the system is only partially symmetric, the corresponding factors may still be as high as 4, 16, and 64.

Section 2.0 presents the method for taking advantage of partial symmetry. Beginning with the pertinent information on the vortex lattice method, it shows how the algebraic system representing a symmetric geometry satisfies certain symmetry conditions. If the geometry is only partially symmetric, the algebraic system can be partitioned so that one portion satisfies the symmetry conditions. The symmetry conditions can be used to produce a quicker solution of the symmetric portion of the system. Section 3.0 summarizes and does an operation count which verifies the value of the method.

In the present domain of application, the number of planes of symmetry, λ, is meaningful only for $\lambda = 1, 2,$ or 3. Mathematically, however, the method generalizes to λ being an arbitrary positive integer. The validity of the method and existence theorems on the inverses involved can then be proved by mathematical induction. This mathematical analysis is presented in the Appendix.

2.0 METHOD

2.1 VORTEX LATTICE METHOD

The vortex lattice method (Ref. 1) models the boundary of a flow regime with N singularities, either vortices or sources. Each singularity induces flow with velocities proportional to its strength. The resultant flow is the superposition of the flows corresponding to all the singularities and an optional free stream. The strengths of the N singularities are determined by imposing boundary conditions at N points called control points. This produces a linear system of N algebraic equations

$$\bar{A}\ \bar{X} = \bar{R} \tag{1}$$

to be solved for the unknown strengths, $\bar{X}$.

The equations for the coefficient matrix, $\bar{A}$, and the right-hand side, $\bar{R}$, are not pertinent to this analysis, but the fact that the coefficient matrix is determined only by the geometry of the boundary is crucial to the development. This fact will be used in Section 2.3 to derive symmetry conditions which are present in $\bar{A}$ if symmetry exists.

Assume that the boundary has a geometrically symmetric portion made up of M singularities and an asymmetric portion made up of p singularities, so

$$N = M + p \tag{2}$$

then the system of equations, Eq. (1), can be partitioned to take the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} R \\ S \end{bmatrix} \tag{3}$$

where A is M x M, B is M x p, C is p x M, D is p x p, X is M x 1, Y is p x 1, R is M x 1, and S is p x 1. Under such a partition, the matrix A can be constructed so that it satisfies the symmetry conditions.

Equation (3) can be written

$$AX + BY = R \tag{4}$$

and

$$CX + DY = S \tag{5}$$

The purpose of the method being presented is to use the symmetry conditions present in A to effect a fast solution of Eq. (4) for X in terms of Y, obtaining

$$X = r - bY \tag{6}$$

Equation (6) can then be substituted into Eq. (5) to obtain

$$(D - Cb)\ Y = (S - Cr) \tag{7}$$

and Gaussian elimination can be used on Eq. (7) to solve for Y. The Y can be substituted into Eq. (6) for the complete solution.

Some discussion is needed on the existence of the solutions. Equation (1) has a solution if $\bar{A}$ is nonsingular. The step from Eq. (4) to Eq. (6) can be taken if A is nonsingular. If $\bar{A}$ and A are nonsingular, then the matrix, (D – Cb), of Eq. (7) is nonsingular. The matrix, $\bar{A}$, will be nonsingular if the system has been modeled correctly. The matrix, A, will be nonsingular if the symmetric portion has been modeled correctly. In general, if a coefficient matrix is singular or even ill-conditioned it is because the system has been improperly modeled.

2.2 NUMBERING THE SINGULARITIES

The vortex model is composed of either vortices or sources commonly called singularities because the corresponding velocity functions are singular at their centers (not to be confused with matrix singularity). The partitioning of the system depends on the ordering of these singularities. The following numbering scheme will be used. Let λ be the number of planes of symmetry, $\lambda = 1, 2,$ or 3. If $\lambda = 1$, then the symmetric portion of the vortex model can be divided into two parts. One part will be called the basis and the other its reflection. If $\lambda = 2$, then the symmetric portion can be divided into four parts, one basis and three reflections. If $\lambda = 3$, there are eight parts with one basis and seven reflections. Let Λ be the number of parts, then

$$\Lambda = 2^{\lambda} \tag{8}$$

Let n be the number of singularities in the basis: then the number of singularities in the symmetric portion is

$$M = \Lambda n \tag{9}$$

Let the planes of symmetry be represented by P_ℓ, where $\ell = 1$ to λ. The singularities of the basis will be numbered from 1 to n. The reflection of the i'th singularity, $1 \leqslant i \leqslant n$, with respect to P_1 will be numbered n + 1. For $\lambda > 1$, the reflection of the i'th singularity, $1 \leqslant i \leqslant 2n$, with respect to P_2 will be numbered 2n + i. For $\lambda > 2$, the reflection of the i'th singularity, $1 \leqslant i \leqslant 4n$ with respect to P_3 will be numbered 4n + i. As already implied by the form of Eq. (3), the nonsymmetric portion will be numbered from M + 1 to N.

With this numbering system, the matrices, A, X, B, and R can be partitioned with respect to the Λ parts

$$A = ((A_{ij})) \qquad X = ((X_i))$$
$$B = ((B_i)) \qquad R = ((R_i)) \tag{10}$$

where i, j = 1 to Λ, A_{ij} is n x n, X_i is n x 1, B_i is n x p, and R_i is n x 1. With this partition, Eq. (4) can be written

$$\sum_{j=1}^{\Lambda} A_{ij} X_j + B_i Y = R_i \tag{11}$$

with i = 1 to Λ. As a special case, if there is no nonsymmetric portion, then Eq. (11) takes the form

$$\sum_{j=1}^{\Lambda} A_{ij} X_j = R_i \tag{12}$$

with i = 1 to Λ and there will be no Eq. (5). This report offers no new approach to the other special case of no symmetric portion, therefore, it will not be considered.

Regarding notation, it will at times be convenient to add a superscript to a variable indicating the number of planes of symmetry, λ. This appendage will in no way alter the definition of the variable.

2.3 SYMMETRY CONDITIONS FOR A

Since A is the coefficient matrix for the symmetric portion and it depends only on the boundary of the system, certain deductions about its internal form can be made. These symmetry conditions will now be derived.

Each part of the symmetric portion has a corresponding part with respect to each plane, the corresponding part being its reflection with respect to that plane. A table can be made of these corresponding parts, which is determined by the numbering system of Sec. 2.2.

Table 1. Corresponding Reflected Parts

Plane ↓	$\lambda = 1$		$\lambda = 2$		$\lambda = 3$			
Part →	1	2	3	4	5	6	7	8
P_1	2	1	4	3	6	5	8	7
P_2	3	4	1	2	7	8	5	6
P_3	5	6	7	8	1	2	3	4

The table for $\lambda = 1$ is above and to the left of the first bold line, and the table for $\lambda = 2$ is above and to the left of the second bold line. The third bold line marks the table for $\lambda = 3$.

A given matrix, A_{ij}, depends only on the geometry of parts i and j and their relative position. Given a plane of symmetry, P_ℓ, if i' and j' are the corresponding parts of i and j with respect to P_ℓ, then parts i' and j' (since they are reflections) have the same geometry and relative positions as do parts i and j. Therefore, $A_{i'j'} = A_{ij}$. Such a relation can be written for each submatrix, A_{ij}, of A with respect to each plane of symmetry, P_ℓ. Thus, there are $\lambda(A)^2$ such relations. For example, when $\lambda = 2$, the relations for the first row of A are

P_1 Relations	P_2 Relations
$A_{11} = A_{22}$	$A_{11} = A_{33}$
$A_{12} = A_{21}$	$A_{12} = A_{34}$
$A_{13} = A_{24}$	$A_{13} = A_{31}$
$A_{14} = A_{23}$	$A_{14} = A_{32}$

For $\lambda = 2$, there are 24 more relations which have not been written out. For a given λ, if all the relations (not all are independent) are written out, A can be written in terms of the submatrices, A_{1j}^{λ}, of the first row. Define

$$A_j = A_{1j} \tag{13}$$

Then the form of A for $\lambda = 1$, 2, and 3 is

$$A^1 = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \tag{14}$$

$$A^2 = \left[\begin{array}{cc|cc} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ \hline A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{array}\right] \tag{15}$$

and

$$A^3 = \left[\begin{array}{cc|cc|cc|cc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ A_2 & A_1 & A_4 & A_3 & A_6 & A_5 & A_8 & A_7 \\ \hline A_3 & A_4 & A_1 & A_2 & A_7 & A_8 & A_5 & A_6 \\ A_4 & A_3 & A_2 & A_1 & A_8 & A_7 & A_6 & A_5 \\ \hline A_5 & A_6 & A_7 & A_8 & A_1 & A_2 & A_3 & A_4 \\ A_6 & A_5 & A_8 & A_7 & A_2 & A_1 & A_4 & A_3 \\ \hline A_7 & A_8 & A_5 & A_6 & A_3 & A_4 & A_1 & A_2 \\ A_8 & A_7 & A_6 & A_5 & A_4 & A_3 & A_2 & A_1 \end{array}\right] \tag{16}$$

A pattern can be noticed in A^k; namely, if A^2 and A^3 are partitioned as shown, then each partition, large or small, has the same form as the right-hand side of Eq. (14). This is the essence of a recursion procedure for generating A^λ in the Appendix.

2.4 SYMMETRY CONDITIONS FOR R

It is not necessary for the boundary conditions to be symmetric to use the solution of Section 2.5. In fact, symmetry of the boundary conditions is of no practical advantage except in the special case, $p = o$. In that case, if R satisfies the symmetry conditions, there is an additional timesaving factor of Λ in the solution.

Boundary conditions, with respect to a plane, can be either symmetric (S-symmetry) or antisymmetric (A-symmetry). As an illustration, for $\lambda = 1$ if a plane is symmetric, then

$$R_2 = R_1$$

and if it is antisymmetric, then

$$R_2 = -R_1$$

Since each plane can be either symmetric or antisymmetric, there are Λ possible combinations. The possibilities are shown in Table 2.

Table 2. Types of Symmetry

Plane	λ = 1		λ = 2		λ = 3			
Type	1	2	3	4	5	6	7	8
P_1	S	A	S	A	S	A	S	A
P_2	S	S	A	A	S	S	A	A
P_3	S	S	S	S	A	A	A	A

The convention for using Table 2 for $\lambda = 1, 2$, or 3 is above and to the left of the indicated bold line, and is the same as explained for Table 1. Table 2 can be used to construct a reflection matrix

$$\eta = ((\eta_{ij})) \tag{17}$$

$i, j = 1$ to Λ. The elements of η are plus or minus ones. The first row consists of plus ones. The second row of η is determined from the first row and the P_1 row of Table 2. An S in Table 2 means copy from the first row, and an A means copy with a change of sign. This process is analogous to the reflection of the basis with respect to P_1. For $\lambda > 1$, the third and fourth rows of η are determined from the first two rows and the P_2 row of Table 2. An S in Table 2 means copy from the first two rows and an A means copy with a change of sign. This is analogous to the reflection of the first two parts with respect to P_2 to obtain parts three and four. For $\lambda > 2$, rows five through eight are obtained from the first four rows and the P_3 row of Table 2 in like manner; this is analogous to the reflection of the first four parts to obtain parts five through eight. By these rules, the following is obtained:

$$\eta^1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{18}$$

$$\eta^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{19}$$

and

$$\eta^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{20}$$

Some useful properties of η are noted. It is symmetric

$$\eta^* = \eta \tag{21}$$

where the asterisk indicates the transpose. Its inverse is given by

$$\eta^{-1} = \frac{1}{\Lambda} \eta \tag{22}$$

This is proved in Theorem 1 of the Appendix. The reflection matrix is an example of a Hadamard matrix, (Refs. 2, 3, and 4). A Hadamard matrix is a square matrix of plus and minus ones such that

$$HH^* = m\ I$$

where m is the order of the matrix.

Having available the reflection matrix, the symmetry conditions for R are simply stated. If the boundary conditions have symmetry of the j'th type, then

$$R_i = \eta_{ij}\ R_1 \tag{23}$$

for i = 1 to Λ.

2.5 SOLUTION

As stated in Section 2.1, the objective of this report is a quick solution of Eq. (4). Sections 2.2, 2.3, and 2.4 were in preparation for this solution. To this end, Eq. (11) is multiplied by η_{ki} and summed with respect to i to obtain

$$\sum_{i=1}^{\Lambda} \sum_{j=1}^{\Lambda} \eta_{ki}\ A_{ij}\ X_j + \sum_{i=1}^{\Lambda} \eta_{ki}\ B_i\ Y = \sum_{i=1}^{\Lambda} \eta_{ki}\ R_i \tag{24}$$

for k = 1 to Λ. Define

$$\alpha_{kj} = \sum_{i=1}^{\Lambda} \eta_{ki}\ A_{ij} \tag{25}$$

$$\theta_k = \frac{1}{\Lambda} \sum_{j=1}^{\Lambda} \eta_{kj}\ X_j \tag{26}$$

$$\beta_k = \frac{1}{\Lambda} \sum_{i=1}^{\Lambda} \eta_{ki}\ B_i \tag{27}$$

$$\rho_k = \frac{1}{\Lambda} \sum_{i=1}^{\Lambda} \eta_{ki}\ R_i \tag{28}$$

and the corresponding matrices

$$\alpha = ((\alpha_{kj})) \qquad \theta = ((\theta_k))$$
$$\beta = ((\beta_k)) \qquad \rho = ((\rho_k)) \tag{29}$$

with k, j = 1 to Λ.

Fundamental to the solution is

$$\alpha_{kj} = \eta_{kj}\ \alpha_{k1} \tag{30}$$

k, j = 1 to Λ. This can be verified by obtaining the product ηA. Except for the sign, the elements of a given row are the same, and the signs of corresponding elements of η and α are the same. The formal proof of Eq. (30) is given in Theorem 3 of the Appendix. Using Eqs. (25) through (30), Eq. (24) becomes

$$\alpha_{k1}\ \theta_k + \beta_k\ Y = \rho_k \tag{31}$$

for k = 1 to Λ. Thus, instead of solving Eq. (4) which has M rows, there are Λ systems of the form of Eq. (31), having n rows each. This can result in a considerable savings, as will be seen in Section 3.0. Equation (31) can be solved for θ_k if α_{k1} is nonsingular. In Theorem 4 of the Appendix, α_{k1}, k = 1 to Λ are shown to be nonsingular if A is nonsingular. Thus, θ_k can be written in terms of Y

$$\theta_k = \overline{r}_k - \overline{b}_k\ Y \tag{32}$$

for k = 1 to Λ. Numerically, it is much more efficient to form the matrix

$$\left[\alpha_{k1} \,\vdots\, \rho_k \,\vdots\, \beta_k\right] \tag{33}$$

and solve by Gaussian elimination than to invert α_{k1}.

Having θ in terms of Y, X can be obtained in terms of Y from Eqs. (26) and (22):

$$X_i = \sum_{k=1}^{\Lambda} \eta_{ik}\ \theta_k \tag{34}$$

i = 1 to Λ. From Eqs. (10), (34), and (32), the r and b of Eq. (6) can be assembled as

$$r = \left(\left(\sum_{k=1}^{\Lambda} \eta_{ik}\ \overline{r}_k\right)\right) \tag{35}$$

and

$$b = \left(\left(\sum_{k=1}^{\Lambda} \eta_{ik} \ \bar{b}_k\right)\right) \tag{36}$$

This completes the solution promised in Section 2.1.

There remains the special case, p = o. Applying the same procedure to Eq. (12) produces

$$a_{ki} \ \theta_k = \rho_k \tag{37}$$

instead of Eq (31) for k = 1 to Λ There is an added benefit if R satisfies the symmetry conditions. Assume the boundary conditions have symmetry of the j'th type, substituting Eq. (23) into (28), and using Eq. (22) gives

$$\rho_k = \delta_{kj} \ R_i \tag{38}$$

k = 1 to Λ, where δ_{kj} is the Kronecker Delta. So ρ_k = o, implying θ_k = o for all k except k = j. Thus, Eq. (37) needs to be solved only for k = j. As a side note, the boundary conditions do not have to be symmetric with respect to all the planes of symmetry to be beneficial. Each plane for which the boundary conditions have either S-symmetry or A-symmetry eliminates half of the systems, Eq. (37).

3.0 SUMMARY

The algorithm can be described as follows. Given $\bar{A}$, $\tilde{\bar{R}}$, λ, n, and p, the following steps are performed.

1. Compute a, Eq. (25).
2. Compute β, Eq. (27).
3. Compute ρ, Eq. (28).
4. For k = 1 to Λ, apply Gaussian elimination to the matrix of Eq. (33).
5. Compute r, Eq. (35).
6. Compute b, Eq. (36).
7. Compute the system of Eq. (7).

8. Solve the system for Y using Gaussian elimination.

9. Compute X, Eq. (6).

The time required to perform this task depends on the number of operations, i.e., additions, subtractions, multiplications, and divisions. Different operations require different times, and even the ratios of times vary with computer. Considering parallel processors, how well a method vectorizes must be taken into account. Such a detailed accounting seemed to be a diversion, so the rudimentary convention of Ref. 5, counting only multiplications and divisions, will be followed.

The first savings from symmetry come in the computation of the coefficient matrix. Equations (14) through (16) show that only Λ of the $(\Lambda)^2$ submatrices of A need to be calculated. Similarly, if R satisfies the symmetry conditions, Eq. (23), only one of the Λ submatrices of R need be calculated.

In the solution, Step 1 requires only additions and subtractions, with η just supplying the signs. The same can be said for Steps 2 and 3, except Step 2 has Mp divisions by Λ, and Step 3 has M divisions by Λ. According to Ref. 5, the number of operations to perform Step 4 is

$$\Lambda \left[\frac{1}{3} n^3 + (p + 1) n^2 - \frac{1}{3} n \right]$$

Steps 5 and 6 require only additions and subtractions. In Step 7, the computation of Cb requires Mp^2 multiplications, and the computation of Cr requires Mp multiplications. Step 8 requires

$$\frac{1}{3} p^3 + p^2 - \frac{1}{3} p$$

operations. Step 9 requires Mp multiplications. The total number of operations, N_1, is

$$\begin{aligned} N_1 = {} & \frac{1}{3} \Lambda n^3 + \Lambda n^2 + \frac{2}{3} \Lambda n \\ & + \Lambda n^2 p + \Lambda n p^2 + 3\Lambda np \\ & + \frac{1}{3} p^3 + p^2 - \frac{1}{3} p \end{aligned} \tag{39}$$

If Eq. (1) were solved directly, the number of operations, N_2, would be

$$N_2 = \frac{1}{3} (M + p)^3 + (M + p)^2 + \frac{1}{3} (M + p)$$

or

$$N_2 = \frac{1}{3}\Lambda^3 n^3 + \Lambda^2 n^2 - \frac{1}{3}\Lambda n + \Lambda^2 n^2 p + \Lambda n p^2 + 2\Lambda n p + \frac{1}{3} p^2 + p^2 - \frac{1}{3} p \tag{40}$$

The number of operations saved is

$$N_2 - N_1 = \frac{1}{3}(\Lambda - 1)\,\Lambda\,(\Lambda + 1)\,n^3 + \Lambda n\left[(\Lambda - 1)\,n - 1\right](p + 1) \tag{41}$$

Equation (41) is a measure of the value of exploiting partial symmetry.

Often, exploiting symmetry is not just a timesaver. Many times it is a matter of whether or not a job can be done; the factor of 8^{λ} pushes it beyond the realm of practicality.

REFERENCES

1. Todd, Donald C. "The AEDC Three-Dimensional, Potential Flow Computer Program." AEDC-TR-75-75, Vol. 1, (AD-A021693), February 1976.

2. Golomb, Solomon W. and Baumert, Leonard D. "The Search for Hadamard Matrices." *The American Mathematical Monthly*, Vol. 70, pp. 12-17, 1963.

3. Pratt, William K., Kane, Julius, and Andrews, Harry C. "Hadamard Transform Image Coding." *Proceedings of the IEEE*, Vol. 57, No. 1, January 1969, pp. 58-68.

4. Fino, Bernard J. and Algozi, Ralph V. "Unified Matrix Treatment of the Fast Walsh-Hadamard Transform." *IEEE Transactions on Computers* Vol. C-25, No. 11, November 1976, pp. 1142-1146.

5. Isaacson, E. and Keller, Herbert Bishop. *Analysis of Numerical Methods*. John Wiley & Sons, Inc., New York, 1966.

APPENDIX A
MATHEMATICAL ANALYSIS

Definition 1: Super Symmetric Block Matrix (SSBM)

Given $n \geqslant 1$, $\lambda \geqslant o$ and

$S = \{\Phi | \Phi$ is an $n \times n$ matrix$\}$

A sequence of sets S_j, $j = o$ to λ can be defined using the following recursive procedure

- $S_o = S$
- $$S_j = \left\{ \Omega | \Omega = \begin{bmatrix} \bar{A} & \hat{A} \\ \hat{A} & \bar{A} \end{bmatrix}, \begin{matrix} \bar{A} \in S_{j-1} \\ \hat{A} \in S_{j-1} \end{matrix} \right\}$$

$j = 1$ to λ. If $A^j \epsilon S_j$, then A^j is called a super symmetric block matrix of order j. It can be noted that

- If $\bar{A}^k$ and $\hat{A}^k$ are SSBM's of order k, then

$$A^{k+1} = \begin{bmatrix} \bar{A}^k & \hat{A}^k \\ \hat{A}^k & \bar{A}^k \end{bmatrix} \tag{A-1}$$

  is an SSBM of order k + 1, and $\bar{A}^k + \hat{A}^k$ and $\bar{A}^k - \hat{A}^k$ are SSBM'S of order k.
- A^1, A^2, and A^3 of Eqs. (14) through (16) are SSBM'S of orders 1, 2, and 3, respectively.

Definition 2. The Reflection Matrix

Given $\lambda \geqslant o$. A sequence of matrices η^j, $j = o$ to λ, can be defined using the following recursive procedure:

$$\eta^o = 1$$

$$\eta^j = \begin{bmatrix} \eta^{j-1} & \eta^{j-1} \\ \eta^{j-1} & -\eta^{j-1} \end{bmatrix} \tag{A-2}$$

$j = 1$ to λ. The matrix η^λ is called the reflection matrix of order λ. It can be verified that η^1, η^2, and η^3 of Eqs. (18) through (20) are the reflection matrices of order 1, 2, and 3, respectively. It is easily proved that the first row and column of a reflection matrix consist of plus ones.

Lemma: Let

$$E = \begin{bmatrix} F & F \\ G & -G \end{bmatrix} \tag{A-3}$$

where F and G are arbitrary n x n matrices. The block matrix, E, is nonsingular if and only if F and G are nonsingular; if nonsingular, then

$$E^{-1} = \frac{1}{2}\begin{bmatrix} F^{-1} & G^{-1} \\ F^{-1} & -G^{-1} \end{bmatrix} \tag{A-4}$$

Proof: Note that

$$\begin{bmatrix} F & F \\ G & -G \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

where I is an n x n identity matrix and 0 is an n x n zero matrix.

Since

$$\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

is nonsingular, E is nonsingular if and only if F and G are nonsingular. If F and G are nonsingular, Eq. (A-4) can be easily verified.

Theorem 1: The reflection matrix of order λ is nonsingular and its inverse is given by

$$\eta^{-1} = \frac{1}{\Lambda}\,\eta \tag{A-5}$$

where $\Lambda = 2^{\lambda}$.

Proof: The theorem is obviously true for $\lambda = o$ since $\Lambda = \eta = 1$ Assume the theorem is true for $\lambda = k$ for some $k \geqslant o$. Applying the Lemma to Eq. (A-2) and substituting Eq. (A-5) into the result produces

$$(\eta^{k+1})^{-1} = \frac{1}{2} \begin{bmatrix} 2^{-k}\,\eta^k & 2^{-k}\,\eta^k \\ 2^{-k}\,\eta^k & -2^{-k}\,\eta^k \end{bmatrix}$$

Factoring out the 2^{-k}, it is seen that the theorem is true for $\lambda = k + 1$ and thus for all λ.

There is occasion in the following analysis to take the product of η and a block matrix. The elements of one are scalars and the elements of the other are n x n matrices. Since the product of a scalar and a matrix is defined, the product of η with a block matrix can be defined in the normal manner.

Theorem 2: A block matrix, A, with n x n blocks is an SSBM of order λ if and only if

$$A = (\eta^{\lambda})^{-1}\ D\ \eta^{\lambda} \tag{A-6}$$

or by Theorem 1

$$A = \frac{1}{\Lambda}\ \eta^{\lambda}\ D\ \eta^{\lambda} \tag{A-7}$$

where D is a block diagonal matrix, that is

$$D = \begin{bmatrix} D_1 & & & 0 \\ & D_2 & & \\ & & \ddots & \\ & 0 & & D_{\Lambda} \end{bmatrix} \tag{A-8}$$

where D_j, $j = 1$ to Λ, are n x n.

Proof: The theorem is obviously true for $\lambda = o$, since $\eta^o = 1$ so $A = D$. Assume the theorem is true for $\lambda = k$ for some $k \geqslant o$. Given an arbitrary D of the form of Eq. (A-8) with $\Lambda = 2^k$ leads to the definition

$$\overline{D} = \mathrm{Diag}\ (D_1, \ldots, D_{\frac{1}{2}\Lambda}) \tag{A-9}$$

and

$$\hat{D} = \text{Diag}\,(D_{\frac{1}{2}\Lambda+1}, \ldots, D_{\Lambda}) \tag{A-10}$$

and Eq. (A-8) can be written

$$D = \begin{bmatrix} \bar{D} & 0 \\ 0 & \hat{D} \end{bmatrix} \tag{A-11}$$

Equation (A-7) then becomes

$$A = \frac{1}{2^{k+1}} \begin{bmatrix} \eta^k & \eta^k \\ \eta^k & -\eta^k \end{bmatrix} \begin{bmatrix} \bar{D} & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} \eta^k & \eta^k \\ \eta^k & -\eta^k \end{bmatrix} \tag{A-12}$$

which can be written

$$A = \begin{bmatrix} \bar{A} & \hat{A} \\ \hat{A} & \bar{A} \end{bmatrix} \tag{A-13}$$

where

$$\bar{A} = \frac{1}{2^k}\, \eta^k \left(\frac{\bar{D} + \hat{D}}{2}\right) \eta^k \tag{A-14}$$

and

$$\hat{A} = \frac{1}{2^k}\, \eta^k \left(\frac{\bar{D} - \hat{D}}{2}\right) \eta^k \tag{A-15}$$

The quantities in parentheses are obviously block diagonal matrices; thus, since the theorem is assumed true for $\lambda = k$, the matrices $\bar{A}$ and $\hat{A}$ are SSBM's of order k. Thus, by the form of Eq. (A-13), the matrix A is an SSBM of order $\lambda = k + 1$.

Assume A is an SSBM of order $\lambda = k + 1$; then A can be written

$$A = \begin{bmatrix} \bar{A} & \hat{A} \\ \hat{A} & \bar{A} \end{bmatrix}$$

where $\bar{A}$ and $\hat{A}$ are SSBM's of order $\lambda = k$. Since the theorem is assumed to be true for $\lambda = k$,

$$\overline{A} = \frac{1}{2^k} \eta^k \; \overline{D} \; \eta^k$$

and

$$\hat{A} = \frac{1}{2^k} \eta^k \; \hat{D} \; \eta^k$$

where $\overline{D}$ and $\hat{D}$ are block diagonal matrices. By matrix algebra, it can be verified that

$$A = \frac{1}{2^{k+1}} \eta^{k+1} \; D \; \eta^{k+1}$$

where

$$D = \begin{bmatrix} \overline{D} + \hat{D} & 0 \\ 0 & \overline{D} - \hat{D} \end{bmatrix}$$

Thus, assuming the truth of the theorem for $\lambda = k$, it is true for $\lambda = k + 1$, and thus for all λ.

Corollary if A is a nonsingular SSBM, then so is its inverse.

Proof: Since A is an SSBM

$$A = \eta^{-1} \; D \; \eta$$

where D is a block diagonal matrix. Its inverse is

$$A^{-1} = \eta^{-1} D^{-1} \; \eta$$

Since D^{-1} is also a block diagonal matrix, A^{-1} is an SSBM.

Theorem 3· Given

$$a = \left((a_{jk})\right) = \eta A \tag{A-16}$$

with j, k = 1 to Λ where A is an SSBM. Then

$$a_{jk} = \eta_{jk} \; a_{j1} \tag{A-17}$$

Proof· Since A is an SSBM, substituting Eq. (A-6) into (A-16) produces

$$a = \eta(\eta^{-1} \; D \; \eta) = D\eta$$

where $D = \text{Diag}(D_1, \ldots, D_\Lambda)$ with D_j, $j = 1$ to Λ, being n x n matrices. Therefore,

$$a_{jk} = (D\eta)_{jk} = D_j \ \eta_{jk} \tag{A-18}$$

when $k = 1$

$$a_{j1} = D_j \ \eta_{j1} = D_j \tag{A-19}$$

since the first column of η consists of plus ones. Subsituting Eq. (A-19) into (A-18) produces the desired result, Eq. (A-17).

Theorem 4: Given

$$a = \eta A \tag{A-20}$$

where A is a nonsingular SSBM, the block matrix a is obviously nonsingular, so let

$$a^{-1} = \frac{1}{\Lambda}\left((\bar{a}_{jk})\right) \tag{A-21}$$

Then,

$$\bar{a}_{jk} = a^{-1}_{kj} \tag{A-22}$$

or by Theorem 3

$$\bar{a}_{jk} = \eta_{jk} \ a^{-1}_{k1} \tag{A-23}$$

Proof: Since A is an SSBM,

$$a = \eta(\eta^{-1} \ D \ \eta) = D \ \eta$$

and

$$a^{-1} = \eta^{-1} \ D^{-1} = \frac{1}{\Lambda} \eta \ D^{-1}$$

Therefore, by the form of Eq. (A-21)

$$\bar{a}_{jk} = (\eta \ D^{-1})_{jk}$$

or

$$\bar{a}_{jk} = \eta_{jk} \ D^{-1}_k \tag{A-24}$$

The desired result, Eq. (A-23) is created when Eq. (A-19) is substituted into Eq. (A-24).

NOMENCLATURE

A	Coefficient matrix for the symmetric portion of the model, Eq. (3)
A_{ij}	Submatrix of A, Eq. (10)
A_j	A_{1j}, Eq. (13)
$\overline{A}$	Coefficient matrix for the whole system, Eq. (1)
B	Submatrix of $\overline{A}$, Eq. (3)
B_i	Submatrix of B, Eq. (10)
b	Eqs. (6) and (36)
$\bar{b}_k$	Eq. (32)
C	Submatrix of $\overline{A}$, Eq. (3)
D	Coefficient matrix for the nonsymmetric portion of the model, Eq. (3). In the Appendix, D is a block diagonal matrix
I	Identity matrix
M	Number of singularities in the symmetric portion of the model, Eq. (9)
N	Number of singularities in the whole model, Eq. (2)
N_1	Operation count using symmetry, Eq. (39)
N_2	Operation count neglecting symmetry, Eq. (40)
n	Number of singularities in the basis of the symmetric portion of the model
P_ℓ	ℓ'th plane of symmetry
p	Number of singularities in the nonsymmetric portion of the model
$\overline{R}$	Right-hand side for the whole system, Eq. (1)
r	Eqs. (6) and (35)
R_i	Right-hand side for Part i, Eq. (10)
$\bar{r}_k$	Eq. (32)
R	Right-hand side for the symmetric portion of the model, Eq. (3)
S	Right-hand side for the nonsymmetric portion of the model, Eq. (3)
X	Strengths of the symmetric portion of the model, Eq. (3)

X_i	Strengths of Part i, Eq. (10)
$\overline{X}$	Strengths of the whole system, Eq. (1)
Y	Strengths of the nonsymmetric portion of the model, Eq. (3)
α	Eq. (29)
α_{kj}	Eq. (25)
β	Eq. (29)
β_k	Eq. (27)
δ_{ij}	Kronecker Delta, Eq. (38)
η	Reflection matrix, Section 2.4
η_{ij}	Elements of η, Eq. (17)
θ	Eq. (29)
θ_k	Eq. (26)
Λ	Number of parts, Eq. (8)
λ	Number of planes of symmetry
ρ	Eq. (29)
ρ_k	Eq. (28)

NOTATION

(( )) Denotes a matrix with the elements indicated

SYMBOLS

*	Denotes the transpose
-1	Denotes the inverse

Any superscript on a variable other than "*" or "-1" does not alter the meaning of the variable, but refers to the number of planes of symmetry, λ.